Energy and angular momentum of the weak gravitational waves on the Schwarzschild background – quasilocal gauge-invariant formulation

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Abstract

It is shown that the axial and polar perturbations of the spherically symmetric black hole can be described in a gauge-invariant way. The reduced phase space describing gravitational waves outside of the horizon is described by the gauge-invariant quantities. Both degrees of freedom fulfill generalized scalar wave equation. For the axial degree of freedom the radial part of the equation corresponds to the Regge-Wheeler result [1] and for the polar one we get Zerilli result [2], see also [4], [5] for both. An important ingredient of the analysis is the *concept of quasilocality* which does duty for the separation of the angular variables in the usual approach. Moreover, there is no need to represent perturbations by normal modes (with time dependence $\exp(-ikt)$), we have fields in spacetime and the Cauchy problem for them is well defined outside of the horizon. The reduced symplectic structure explains the origin of the axial and polar invariants. It allows to introduce an energy and angular momentum for the gravitational waves which is invariant with respect to the gauge transformations. Both generators represent quadratic approximation of the ADM nonlinear formulae in terms of the perturbations of the Schwarzschild metric. We also discuss the boundary-initial value problem for the linearized Einstein equations on a Schwarzschild background outside of the horizon.

1 Introduction

In section 2 we introduce standard notions for linearized gravity and its 3+1 formulation. In the next section the analysis of constraints and Killing fields on an initial surface gives charges for the linear field on the Schwarzschild background.

Section 4 contains the main technical results related to the so-called 2+1 decomposition of the initial data and to the gauge invariant description of the evolution. The invariants introduced in this section contain the full gauge-independent information about initial data. If we "insert" the initial value constraints into the canonical symplectic structure we can express the symplectic 2-form Ω in terms of the invariants. This is precisely shown in Appendix B. This way our invariants play a role of the reduced unconstrained initial data which is gauge independent. Moreover, the axial degree of freedom fulfills four-dimensional counterpart of the radial equation proposed by Regge and Wheeler [1], [4], [5] and the polar invariant is related to the Zerilli equation [2], [4], [5].

Section 5 is devoted to the stationary solutions, their behaviour on the horizon and precise interpretation of the "mono-dipole" solutions. In particular, the result of Vishveshwara [3] that stationary

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axial perturbation can exist only for dipole perturbation can be easily recovered. However, if we assume more general conditions on a horizon we may have other solutions. The dipole part of the axial invariant corresponds to the spherical symmetry of the background metric and represents angular momentum. On the other hand, there is no dipole invariants in the polar part because the background metric is not invariant with respect to the boosts and spatial translations[1]. Moreover, the lack of invariant in the dipole polar part corresponds to the fact that this part of the metric can be always "gauged away". We would like to stress that mono-dipole perturbations of the Schwarzschild metric represent different phenomena than the higher-pole perturbations. It is clear from the approach that $l \geq 2$ represents *gravitational wave perturbation* whereas $l = 0$ and $l = 1$ correspond to the *charges*. For example, $l = 0$ in polar part represents infinitesimal perturbation of the mass (we move in the space of solutions from one Schwarzschild solution to another), dipole part in axial part corresponds to the infinitesimal angular momentum (we move from Schwarzschild to Kerr solution). Formally also monopole part in axial degree of freedom can be analyzed and represents Taub-NUT charge but this move is excluded by topology.

Reduction of the symplectic form presented in Appendix B allows to introduce invariants from the symplectic point of view in section 6. In the next section we define (in a gauge-invariant way) hamiltonian system[2], energy and angular momentum generators and boundary-initial value problem for the linearized field on the Schwarzschild background. Moreover, we are able to show that the obtained hamiltonian is a quadratic approximation of the ADM mass defined at spatial infinity for the full nonlinear Einstein theory and this particular result will be described elsewhere.

The energy and angular momentum generators are well defined for regular radiation data $\underline{\mathbf{x}}$, $\underline{\mathbf{X}}$, $\underline{\mathbf{y}}$, $\underline{\mathbf{Y}}$ which is finite on the horizon and vanishing at spatial infinity according to the S.A.F. Christodolou-Kleinerman condition [14].

In Appendix A we show how to reconstruct the full four-metric $h_{\mu\nu}$ from the invariants assuming the gauge conditions used in [1] and [3]. This construction explains the relations between our invariants and the special form of the metric used in [1] and [3]. In particular, we examine precisely "mono-dipole" part of the metric ($l = 0$ and $l = 1$) which seems to be not fully analyzed in literature.

2 Linearized gravity

In this section we remind some standard notions related to the Einstein equations and the initial value problem.

Linearized Einstein theory (see e.g. [8] or [9]) can be formulated as follows. Einstein equation

$$2\mathrm{G}_{\mu\nu}(g) = 16\pi\mathrm{T}_{\mu\nu} \tag{2.1}$$

after linearization gives

$$h_{\mu\alpha;\nu}{}^{;\alpha} + h_{\nu\alpha;\mu}{}^{;\alpha} - h_{\mu\nu}{}^{;\alpha}{}_{\alpha} - (\eta^{\alpha\beta}h_{\alpha\beta})_{;\mu\nu} - \eta_{\mu\nu}[h_{\alpha\beta}{}^{;\alpha\beta} - h_{\alpha}{}^{\alpha;\beta}{}_{\beta}] = 16\pi T_{\mu\nu} \tag{2.2}$$

where pseudoriemannian metric $\mathrm{g}_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, $\eta_{\mu\nu}$ is the background metric and ";" denotes four–dimensional covariant derivative with respect to the metric $\eta_{\mu\nu}$. Moreover, we assumed that $\eta_{\mu\nu}$ is a vacuum solution of Einstein equation ($\mathrm{G}_{\mu\nu}(\eta) = 0$).

The (3+1)-decomposition of (2.2) gives 6 dynamical equations for the space–space components h_{kl} of the metric (latin indices run from 1 to 3) and 4 equations which do not contain time derivatives of h_{kl}. It is possible to introduce this decomposition straightforward. However, a natural way for the formulation of the (3+1)-splitting for the equation (2.2) is to linearize ADM formulation of the initial value problem for the full nonlinear Einstein equation (2.1). We shall introduce canonical variables for the linearized case. They appear in a natural way if we start from the ADM formulation of the initial value problem for Einstein equations [7].

[1]Dipole invariants exist in flat Minkowski space and they represent linear momentum and center of mass respectively (see [19]).

[2]After separation of the angular variables several "hamiltonian" results proposed by the author can be easily translated into Moncrief's approach presented in [5].

Let $(\mathrm{g}_{kl}, \mathrm{P}^{kl})$ be the Cauchy data for Einstein equations on a three–dimensional space-like surface Σ. This means that g_{kl} is a Riemannian metric on Σ and P^{kl} is a symmetric tensor density, which we identify with the ADM momentum [7], i.e.

$$\mathrm{P}^{kl} = \sqrt{\det \mathrm{g}_{mn}}(\mathrm{g}^{kl}\mathrm{Tr}K - K^{kl})$$

where K_{kl} is the second fundamental form (external curvature) of the imbedding of Σ into a spacetime M.
The 12 functions $(\mathrm{g}_{kl}, \mathrm{P}^{kl})$ must fulfill 4 Gauss–Codazzi constraints

$$\mathrm{P}_i{}^l{}_{|l} = 8\pi\sqrt{\det g_{mn}}\, \mathrm{T}_{i\mu}n^\mu \tag{2.3}$$

$$(\det \mathrm{g}_{mn})\mathrm{R} - \mathrm{P}^{kl}\mathrm{P}_{kl} + \frac{1}{2}(\mathrm{P}^{kl}g_{kl})^2 = 16\pi(\det \mathrm{g}_{mn})\mathrm{T}_{\mu\nu}n^\mu n^\nu \tag{2.4}$$

where $\mathrm{T}_{\mu\nu}$ is an energy momentum tensor of the matter, by R we denote the (three–dimensional) scalar curvature of g_{kl}, n^μ is a future timelike four–vector normal to the hypersurface Σ and the calculations have been made with respect to the three–metric g_{kl} ("$|$" denotes the covariant derivative, indices are raised and lowered etc.).

The Einstein equations and the definition of the metric connection imply the first order (in time) differential equations for g_{kl} and P^{kl} (see [7] or [8] p. 525) and contain the lapse function N and the shift vector N^k as parameters

$$\dot{\mathrm{g}}_{kl} = \frac{2N}{\sqrt{\mathrm{g}}}\left(\mathrm{P}_{kl} - \frac{1}{2}\mathrm{g}_{kl}\mathrm{P}\right) + N_{k|l} + N_{l|k} \tag{2.5}$$

where $\mathrm{g} := \det \mathrm{g}_{mn}$ and $\mathrm{P} := \mathrm{P}^{kl}\mathrm{g}_{kl}$

$$\begin{aligned}\dot{\mathrm{P}}^{kl} &= -N\sqrt{\mathrm{g}}\mathrm{R}^{kl} + \sqrt{\mathrm{g}}\left(N^{|kl} - \mathrm{g}^{kl}N^{|m}{}_{|m}\right) + \\ &+\frac{1}{2}N\sqrt{\mathrm{g}}\mathrm{g}^{kl}\mathrm{R} - \frac{2N}{\sqrt{\mathrm{g}}}\left(\mathrm{P}^{km}\mathrm{P}_m{}^l - \frac{1}{2}\mathrm{P}\mathrm{P}^{kl}\right) + \left(\mathrm{P}^{kl}N^m\right)_{|m} + \\ &+ \frac{N}{2\sqrt{\mathrm{g}}}\mathrm{g}^{kl}\left(\mathrm{P}^{kl}\mathrm{P}_{kl} - \frac{1}{2}\mathrm{P}^2\right) - N^k{}_{|m}\mathrm{P}^{ml} - N^l{}_{|m}\mathrm{P}^{mk} + 8\pi N\sqrt{\mathrm{g}}\mathrm{T}_{mn}\mathrm{g}^{km}\mathrm{g}^{ln}\end{aligned} \tag{2.6}$$

Let us consider an initial value problem for the linearized Einstein equations on Schwarzschild background $\eta_{\mu\nu}$:

$$\eta_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu = -\left(1 - \frac{2m}{r}\right)\mathrm{d}t^2 + \left(1 - \frac{2m}{r}\right)^{-1}\mathrm{d}r^2 + r^2\mathrm{d}\theta^2 + r^2\sin^2\theta\mathrm{d}\varphi^2 \tag{2.7}$$

together with the radial coordinates: $x^3 = r$, $x^1 = \theta$, $x^2 = \varphi$. Moreover, $t = x^0$ denotes the time coordinate. We consider only the part of the Schwarzschild spacetime outside of the horizon, $r \geq 2m$.

We use the following convention for indices: greek indices $\mu, \nu, \ldots$ run from 0 to 3; $k, l, \ldots$ are spatial coordinates and run from 1 to 3; $A, B, \ldots$ are spherical angles (θ, φ) on a two-dimensional sphere $S(r) := \{r = x^3 = \mathrm{const}\}$ and run from 1 to 2. Moreover, let η_{AB} denote a two-dimensional metric on $S(r)$.

Let $v := 1 - \frac{2m}{r}$. There are the following non-vanishing Christoffel symbols for the metric (2.7):

$$\Gamma^3{}_{33} = -\frac{m}{vr^2}\,;\;\; \Gamma^3{}_{AB} = -\frac{v}{r}\eta_{AB}\,;\;\; \Gamma^A{}_{3B} = \frac{1}{r}\delta^A{}_B\,;\;\; \Gamma^3{}_{00} = \frac{mv}{r^2}\,;\;\; \Gamma^0{}_{30} = \frac{m}{vr^2}\,;\;\; \Gamma^A{}_{BC}$$

where $\delta^A{}_B$ is the Kronecker's symbol and $\Gamma^A{}_{BC}$ are the same as for a standard unit sphere $S(1)$ (in usual spherical coordinates $\Gamma^\theta{}_{\phi\phi} = -\sin\theta\cos\theta$ and $\Gamma^\phi{}_{\phi\theta} = \cot\theta$).

The curvature of the background metric we denote by $C^{\mu}{}_{\nu\lambda\kappa}$ and the following components of the Riemann tensor are non-vanishing (up to the symmetries of the indices)

$$C^0{}_{A0B} = C^3{}_{A3B} = \frac{m}{r^3}\eta_{AB}\,; \quad C^{AB}{}_{CD} = \frac{2m}{r^3}\left(\delta^A{}_C\delta^B{}_D - \delta^A{}_D\delta^B{}_C\right)$$

We can introduce the following submanifolds of the Schwarzschild spacetime M:

$$\Sigma_s := \{x \in M \;:\; x^0 = s, x^3 \geq 2m\} = \bigcup_{r\in[2m,\infty[} S_s(r) \quad \text{where} \;\; S_s(r) := \{x \in \Sigma_s \;:\; x^3 = r\} \tag{2.8}$$

and Σ_s is a partial Cauchy surface outside of the horizon.

The surface Σ carries the induced Riemannian metric η_{kl}:

$$\eta_{kl}\mathrm{d}y^k\mathrm{d}y^l = \frac{1}{v}\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2) \tag{2.9}$$

Usually it is convenient to change the coordinate r to r_* which is a solution of an ordinary differential equation (see [4]):

$$\frac{\mathrm{d}r_*}{\mathrm{d}r} = \frac{1}{v}\,; \quad r_* := r + 2m\ln\left(\frac{r}{2m} - 1\right) \tag{2.10}$$

and moves horizon to $-\infty$.

The ADM momentum P^{kl} for the metric (2.9) on each slice Σ_s vanishes ($\mathrm{P}^{kl} = 0$). The shift vector is also trivial ($N^k = 0$) and the lapse $N = \sqrt{v}$ is vanishing on the horizon. Moreover, the Ricci tensor for the three-metric η_{kl} has the following components:

$$\mathrm{R}_{kl} = \frac{1}{N}N_{|kl}$$

and the scalar curvature R vanishes.

Let us define the linearized variations (h_{kl}, P^{kl}) of the full nonlinear Cauchy data $(\mathrm{g}_{kl}, \mathrm{P}^{kl})$ around background data $(\eta_{kl}, 0)$

$$h_{kl} := \mathrm{g}_{kl} - \eta_{kl}\,, \quad P^{kl} := \mathrm{P}^{kl} \tag{2.11}$$

We should now rewrite equations (2.3)–(2.6) in a linearized form in terms of (h_{kl}, P^{kl}). Let us denote $P := \eta_{kl}P^{kl}$ and $h := \eta^{kl}h_{kl}$. The vector constraint (2.3) can be linearized as follows

$$\mathrm{P}_i{}^l{}_{|l} \approx P_i{}^l{}_{|l} \tag{2.12}$$

Let us stress that the symbol "|" has different meanings on the left-hand side and on the right-hand side of the above formula. It denotes the covariant derivative with respect to the full nonlinear metric g_{kl} when applied to the ADM momentum P^{kl}, but on the right-hand side it means the covariant derivative with respect to the background metric η_{kl}. The scalar constraint (2.4) after linearization takes the form

$$\sqrt{\mathrm{g}}\mathrm{R} - \frac{1}{\sqrt{\mathrm{g}}}\left(\mathrm{P}^{kl}\mathrm{P}_{kl} - \frac{1}{2}(\mathrm{P}^{kl}\mathrm{g}_{kl})^2\right) \approx \sqrt{\eta}\left[\left(h^{kl}{}_{|l} - h^{|k}\right)_{|k} - h^{kl}\mathrm{R}_{kl}\right] \tag{2.13}$$

where $\sqrt{\eta} := \sqrt{\det\eta_{kl}}$.

The linearized constraints for the vacuum ($T_{\mu\nu} = 0$) have the form

$$P_l{}^k{}_{|k} = 0 \tag{2.14}$$

$$\left(h^{kl}{}_{|l} - h^{|k}\right)_{|k} - h^{kl}\mathrm{R}_{kl} = 0 \tag{2.15}$$

The linearization of (2.5) leads to the equation

$$\dot{h}_{kl} \;=\; \frac{2N}{\sqrt{\eta}}\left(P_{kl} - \frac{1}{2}\eta_{kl}P\right) + h_{0k|l} + h_{0l|k} \tag{2.16}$$

where $N := \frac{1}{\sqrt{-\eta^{00}}} = \sqrt{v}$, $N_k = \eta_{0k} = 0$ are the lapse and shift for the background. Let us denote the linearized lapse by $n := \frac{1}{2}\sqrt{v}h^0_0$. The linearization of (2.6) takes the form

$$\frac{1}{\sqrt{\eta}}\dot{P}_{kl} = -n\mathrm{R}_{kl} - N\delta R_{kl} + n_{|kl} - N_m\delta\Gamma^m{}_{kl} - \eta_{kl}\left(n^{|m}{}_{|m} - \eta^{ij}\delta\Gamma^m{}_{ij}N_{|m} - h^{mn}N_{|mn}\right) \quad (2.17)$$

where

$$\delta\Gamma^m{}_{kl} := \frac{1}{2}\left(h^m{}_{k|l} + h^m{}_{l|k} - h_{ml}{}^{|k}\right)$$

is the linearized Christoffel symbol and similarly

$$\delta R_{kl} := \frac{1}{2}\left(h^m{}_{k|lm} + h^m{}_{l|km} - h_{kl|m}{}^m - h_{|kl}\right)$$

is the linearized Ricci tensor.

It is well known (see for example [19]) that the linearized Einstein equations (2.2) are invariant with respect to the "gauge" transformation:

$$h_{\mu\nu} \to h_{\mu\nu} + \xi_{\mu;\nu} + \xi_{\nu;\mu} \quad (2.18)$$

where ξ_μ is a covector field. The (3+1)-decomposition of the gauge acts on the Cauchy data in the following way

$$2v\Lambda^{-1}P_{kl} \quad \to \quad 2v\Lambda^{-1}P_{kl} + (v\xi^0_{|k})_{|l} + (v\xi^0_{|l})_{|k} - \eta_{kl}(v\xi^{0|m})_m \quad (2.19)$$

$$h_{kl} \quad \to \quad h_{kl} + \xi_{l|k} + \xi_{k|l} \quad (2.20)$$

where $\Lambda := \sqrt{v}\sqrt{\det\eta_{kl}}$ $(= r^2\sin\theta)$. It can be easily checked that the scalar constraint (2.15) and the vector constraint (2.14) are invariant with respect to the gauge transformations (2.19) and (2.20). The Cauchy data $(h_{kl},\, P^{kl})$ and $(\overline{h}_{kl},\, \overline{P}^{kl})$ on Σ are equivalent to each other if they can be related by the gauge transformation ξ_μ. The evolution of canonical variables P^{kl} and h_{kl} given by equations (2.16), (2.17) is not unique unless the lapse function n and the shift vector h_{0k} are specified.

We will show in the sequel that it is possible to define a reduced dynamics in terms of invariants, which is no longer sensitive on gauge conditions. The construction is analogous to the analysis given in [16].

3 "Charges" on the Schwarzschild background

The vector constraint (2.14) allows to introduce "charges" related to the symmetries of the background metric. There are three generators of the rotation group, which are simultaneously Killing vectors on the initial surface Σ. Let us denote this Killing field by Z^k. It is a solution of the Killing equation

$$Z_{k|l} + Z_{l|k} = 0 \quad (3.1)$$

Let $V \subset \Sigma$ be a compact region in Σ. For example $V := \bigcup_{r\in[r_0,r_1]} S_s(r)$ and $\partial V = S_s(r_0)\cup S_s(r_1)$. From (2.14) and (3.1) we get

$$0 = \int_V P^{kl}{}_{|l}Z_k = \int_V (P^{kl}Z_k)_{|l} = \int_{\partial V} P^{3k}Z_k \quad (3.2)$$

The equation (3.2) expresses the "Gauss" law for the angular momentum charge "measured" by the flux integral. It is easy to relate this charge to the dipole part of invariant $\mathbf{y}$, which will appear in the sequel – (4.19) in the next section. For example, when $Z = \partial/\partial\phi$ we have

$$16\pi s^z := 16\pi j^{xy} = -2\int_{\partial V} P^3{}_\phi = -2\int_{\partial V} P^3{}_A(r^2\varepsilon^{AB}\cos\theta)_{||B} =$$

$$= 2\int_{\partial V} r^2 P^3{}_{A||B}\varepsilon^{AB}\cos\theta = \int_{\partial V}\Lambda\mathbf{y}\cos\theta \tag{3.3}$$

The time translation defines a mass charge from the scalar constraint (2.15) as follows

$$0 = \int_V N\sqrt{\eta}\left[\left(h^{kl}{}_{|l} - h^{|k}\right)_{|k} - h^{kl}N_{|kl}\right] =$$

$$= \int_V \left[N\sqrt{\eta}\left(h^{lk}{}_{|k} - h^{|l}\right) + \sqrt{\eta}\left(N^{|l}h - N_{|k}h^{kl}\right)\right]_{|l} =$$

$$= \int_{\partial V}\Lambda\left(h^{3k}{}_{|k} - h^{|3} + \frac{1}{N}hN^{|3} - \frac{1}{N}h^{k3}N_{|k}\right) \tag{3.4}$$

and it can be related to the monopole part of an invariant $\mathbf{x}$ introduced in the next section by (4.21).

$$16\pi p^0 = \int_{\partial V}\Lambda\left(Nh^{3k}{}_{|k} - Nh^{|3} + N_k h^{k3} - N^3 h\right) =$$

$$= \int_{\partial V}\frac{\Lambda}{r}\left(2h^{33} - vrH,_3 - (1 - \frac{3m}{r})H\right) = \int_{\partial V}\frac{\Lambda}{r}\mathcal{B}^{-1}\mathbf{x} \tag{3.5}$$

4 Equations of motion for the invariants

The radial foliation of the Cauchy surface Σ related to the spherical symmetry allows to perform (2+1)-decomposition of the initial data. In this section we introduce reduced gauge invariant data on Σ for the gravitational field, similar to the invariants introduced in [16]. For this purpose we use a spherical foliation of Σ (see formulae (2.8) and (2.9)).

Let $\overset{\circ}{\Delta}$ denotes the two–dimensional Laplace–Beltrami operator on a unit sphere $S(1)$. Moreover, $H := \eta^{AB}h_{AB}$, $\chi_{AB} := h_{AB} - \frac{1}{2}\eta_{AB}H$, $S := \eta^{AB}P_{AB}$, $S_{AB} := P_{AB} - \frac{1}{2}\eta_{AB}S$ according to the notation used in [16].

The spatial gauge (2.20) splits in the following way

$$h_{33} \quad\to\quad h_{33} + \frac{2}{\sqrt{v}}(\sqrt{v}\xi_3)_{,3} \tag{4.1}$$

$$h_{3A} \quad\to\quad h_{3A} + \xi_{3,A} + \xi_{A,3} - \frac{2}{r}\xi_A \tag{4.2}$$

$$h_{AB} \quad\to\quad h_{AB} + \xi_{A||B} + \xi_{B||A} + \frac{2}{r}\eta_{AB}\xi^3 \tag{4.3}$$

where by "$||$" we denote the covariant derivative with respect to the two–metric η_{AB} on $S(r)$. Similarly, the temporal gauge (2.19) can be splitted as follows

$$\Lambda^{-1}P^3{}_3 \quad\to\quad \Lambda^{-1}P^3{}_3 - \xi^{0||A}{}_A - \frac{2}{r}\xi^{0,3} \tag{4.4}$$

$$\Lambda^{-1}P_{3A} \quad\to\quad \Lambda^{-1}P_{3A} + \left(\xi^0_{,3} - \frac{1}{r}\xi^0 + \frac{m}{vr^2}\xi^0\right)_{||A} \tag{4.5}$$

$$\Lambda^{-1}S_{AB} \quad\to\quad \Lambda^{-1}S_{AB} + \xi^0_{||AB} - \frac{1}{2}\eta_{AB}\xi^{0||C}{}_C \tag{4.6}$$

$$\Lambda^{-1}S \quad\to\quad \Lambda^{-1}S - \frac{2}{N}(N\xi^{0,3})_{,3} - \frac{2}{r}\xi^{0,3} - \xi^{0||C}{}_C \tag{4.7}$$

It is also quite easy to rewrite the (2+1)-decomposition of (2.16)

$$\dot{h}_{33} = \Lambda^{-1}(P^3{}_3 - S) + \frac{2}{\sqrt{v}}(\sqrt{v}h_{03}),_3 \tag{4.8}$$

$$\dot{h}_{3A} = 2v\Lambda^{-1}P_{3A} + h_{03||A} + h_{0A\,,\,3} - \frac{2}{r}h_{0A} \tag{4.9}$$

$$\dot{h}_{AB} = 2v\Lambda^{-1}S_{AB} - \eta_{AB}\Lambda^{-1}P^{33} + h_{0A||B} + h_{0B||A} + 2vr^{-1}\eta_{AB}h_{03} \tag{4.10}$$

The dynamical equations (2.17) take the following (2+1)-form:

$$2\Lambda^{-1}\dot{P}_{33} = -\frac{1}{v}h^0{}_0{}^{||A}{}_A - 2r^{-1}h^0{}_{0\,,\,3} + \frac{1}{v}h_3^{3||A}{}_A + 2r^{-2}h^3{}_3 +$$

$$+ (H_{,\,3} - 2h_3{}^A{}_{||A} - 2r^{-1}h_3{}^3)_{,\,3} + 2r^{-1}(H_{,\,3} - 2h_3{}^A{}_{||A} - 2r^{-1}h_3{}^3) \tag{4.11}$$

$$2\Lambda^{-1}\dot{P}_{3C} = \left[\frac{1}{\sqrt{v}}(\sqrt{v}h^0{}_0)_{,\,3} - r^{-1}h^0{}_0\right]_{||C} - \frac{m}{vr^2}h^3_{3||C} + \frac{1}{2}(H_{,\,3} - 2h_3{}^A{}_{||A} - 2r^{-1}h^3{}_3)_{||C} +$$

$$+ h_{3C||A}{}^{||A} + \frac{1}{r^2}h_{3C} - \chi^A{}_{C||A\,,\,3} \tag{4.12}$$

$$2\Lambda^{-1}\dot{S}_{AB} = h^0{}_{0||AB} - \frac{1}{2}\eta_{AB}h^0{}_0{}^{||C}{}_C + h^3{}_{3||AB} - \frac{1}{2}\eta_{AB}h^3{}_3{}^{||C}{}_C +$$

$$-(h^3{}_{A||B} + h^3{}_{B||A} - \eta_{AB}h^{3C}{}_{||C})_{,\,3} + (v\chi^C{}_{B\,,\,3}\eta_{CA})_{,\,3} +$$

$$+ \chi_{AB}{}^{||C}{}_{||C} - \chi^C{}_{A||BC} - \chi^C{}_{B||AC} + \eta_{AB}\chi^{CD}{}_{||CD} + \frac{2}{r^2}\chi_{AB} \tag{4.13}$$

$$2\Lambda^{-1}\dot{S} = -\sqrt{v}(\sqrt{v}h^0_{0,3})_{,3} - \frac{1}{r}h^0_{0,3} - h_0^{0||A}{}_A + (h^3{}_3 + H)^{||A}{}_A + \frac{2}{r^2}(h^3{}_3 + H) - \frac{12m}{r^3}h^3{}_3 +$$

$$+ \frac{2m}{r^2}h^3{}_{3,3} + v(H_{,\,3} - 2h_3{}^A{}_{||A} - 2r^{-1}h_3{}^3)_{,\,3} + 4\frac{v}{r}(H_{,\,3} - 2h_3{}^A{}_{||A} - 2r^{-1}h_3{}^3) - 2\chi^{CD}{}_{||CD} \tag{4.14}$$

The vector constraint (2.14) splits in a similar way

$$\frac{1}{\sqrt{v}}(\sqrt{v}P^3{}_3)_{,\,3} + P_3{}^A{}_{||A} - r^{-1}S = 0 \tag{4.15}$$

$$P^3{}_{A\,,\,3} + S_A{}^B{}_{||B} + \frac{1}{2}S_{||A} = 0 \tag{4.16}$$

And finally the (2+1)-decomposition of the scalar constraint (2.15) can be written in the form

$$h^{|l}{}_l - h^{kl}{}_{|kl} + h^{kl}\mathrm{R}_{kl} = \frac{\sqrt{v}}{r^3}\left[r^2\sqrt{v}(rH_{,\,3} - 2rh_{3A}{}^{||A} - 2h_3{}^3)\right]_{,\,3} +$$

$$+ h^3{}_{3||A}{}^A + 2r^{-2}h^3{}_3 - \frac{6m}{r^3}h^3{}_3 + \frac{1}{2}H^{||C}{}_C + r^{-2}H - \chi^{AB}{}_{||AB} = 0 \tag{4.17}$$

Let us notice that we can split the dynamics into two separate parts which we call axial and polar respectively. The axial part of initial data consists of two momentum components $P^{3A||B}\varepsilon_{AB}$, $S^C{}_{A||CB}\varepsilon^{AB}$ and two metric components $h_{3A||B}\varepsilon^{AB}$, $\chi^C{}_{A||CB}\varepsilon^{AB}$. The only gauge freedom is contained in $\xi_{A||B}\varepsilon^{AB}$ which acts on the metric components. The gauge invariants $P^{3A||B}\varepsilon_{AB}$ and $S^C{}_{A||CB}\varepsilon^{AB}$ are related by the curl part of the vector constraint (4.16) as follows:

$$(r^2P^{3A||B}\varepsilon_{AB})_{,\,3} + r^2S^C{}_{A||CB}\varepsilon^{AB} = 0 \tag{4.18}$$

It is easy to verify that the following pair of gauge invariants:

$$\mathbf{y} := 2\Lambda^{-1}r^2P^{3A||B}\varepsilon_{AB} \tag{4.19}$$

$$\mathbf{Y} := \Lambda(\overset{\circ}{\Delta} + 2) h_{3A||B}\varepsilon^{AB} - r^2(\Lambda \chi^C{}_{A||CB}\varepsilon^{AB}),_3 \tag{4.20}$$

contains the whole information about axial part of initial data up to the gauge freedom (see also Appendix A). We will show in the sequel that this is a canonical pair with respect to the symplectic structure of linearized Einstein equations.

Let us define the polar invariants as follows

$$\mathbf{x} := r^2\chi^{AB}{}_{||AB} - \frac{1}{2}(\overset{\circ}{\Delta} + 2)H + \mathcal{B}\left[2h^{33} + 2rh^{3C}{}_{||C} - rvH,_3\right] \tag{4.21}$$

$$\mathbf{X} := 2r^2 S^{AB}{}_{||AB} + \mathcal{B}\left[2rP^{3A}{}_{||A} + \overset{\circ}{\Delta} P^3{}_3\right] \tag{4.22}$$

where

$$\mathcal{B} := (\overset{\circ}{\Delta} + 2)\left(\overset{\circ}{\Delta} + 2 - \frac{6m}{r}\right)^{-1}$$

is a *quasilocal* operator – it is local with respect to the coordinates t, r but non-local on each sphere $S(r)$. The proof that $\mathbf{x}$ and $\mathbf{X}$ contain the whole information about polar part of initial data $P^{3A}{}_{||A}$, $S^{AB}{}_{||AB}$, $P^3{}_3$, S, $h^{3C}{}_{||C}$, $\chi^{AB}{}_{||AB}$, $h^3{}_3$, H up to the gauge transformation $\xi^0, \xi^3, \xi^A{}_{||A}$ we give in Appendix A. Moreover, in Appendix B we show that $(\mathbf{x}, \mathbf{X}, \mathbf{y}, \mathbf{Y})$ is the reduced canonical initial data on Σ.

We can check the reduced field equations for the axial invariants

$$\dot{\mathbf{y}} = \frac{v}{\Lambda}\mathbf{Y} \tag{4.23}$$

$$\dot{\mathbf{Y}} = \Lambda\left\{\left[\frac{v}{r^2}(r^2\mathbf{y}),_3\right],_3 + \frac{1}{r^2}(\overset{\circ}{\Delta} + 2)\mathbf{y}\right\} \tag{4.24}$$

More precisely, the curl part of (4.12) gives (4.23) and (4.24) follows directly from (4.9), (4.10) and may be checked by inspection.

It can be easily verified that the axial invariant $\mathbf{y}$ fulfills the generalized scalar wave equation (as a consequence of the above dynamical equations)

$$\left(\Box + \frac{8m}{r^3}\right)\mathbf{y} = 0 \tag{4.25}$$

where $\Box$ denotes the usual wave operator with respect to the background metric $\eta_{\mu\nu}$.

There exists a simple relation between the equation (4.25) and the so-called *Regge-Wheeler equation* [1], [4], [5]. Let us rewrite equation (4.25) in the following form:

$$-\ddot{\mathbf{y}} + \frac{v}{r}\left[v(r\mathbf{y}),_3\right],_3 = V^{(-)}\mathbf{y} \tag{4.26}$$

where $V^{(-)}$ is a "spherical operator" defined as follows:

$$V^{(-)} := -\frac{v}{r^2}\left(\overset{\circ}{\Delta} + \frac{6m}{r}\right)$$

If we insert the invariant $\mathbf{y}$ in a special form $\mathbf{y} = \exp(i\sigma t)Y_l(\theta,\phi)Z^{(-)}(r)/r$ into equation (4.26), we obtain Regge-Wheeler equation

$$\left(\frac{d^2}{dr_*^2} + \sigma^2\right)Z^{(-)} = V^{(-)}Z^{(-)}$$

Here Y_l is a spherical harmonics such that $\overset{\circ}{\Delta} Y_l = -l(l+1)Y_l$ and coordinate r_* is defined by (2.10).

The polar invariants fulfill the following equations:

$$\dot{\mathbf{x}} = \frac{v}{\Lambda}\mathbf{X} \tag{4.27}$$

$$\dot{\mathbf{X}} = \frac{\Lambda}{r^2}\left\{\left(vr^2\mathbf{x},_3\right),_3 + \left[\overset{\circ}{\Delta} + v(1-2\mathcal{B}) + 1\right]\mathcal{B}\mathbf{x}\right\} \tag{4.28}$$

To obtain (4.27) we need equations (4.8), (4.9), (4.10) and vector constraints (4.15), (4.16). Similarly, (4.28) is a consequence of (4.11-4.13) and scalar constraint (4.17) (see also Appendix A).

There exists also a generalized scalar wave equation for the polar invariant but it is no longer local, it is only quasilocal:

$$\left[\Box + \frac{8m}{r^3}(\overset{\circ}{\Delta} - 1)\left(\overset{\circ}{\Delta} + 2 - \frac{3m}{r}\right)\left(\overset{\circ}{\Delta} + 2 - \frac{6m}{r}\right)^{-2}\right]\mathbf{x} = 0 \tag{4.29}$$

The similar equation to the (4.26) can be presented in the analogous form

$$-\ddot{\mathbf{x}} + \frac{v}{r}\left[v(r\mathbf{x})_{,3}\right]_{,3} = V^{(+)}\mathbf{x} \tag{4.30}$$

but now the operator $V^{(+)}$ is defined "quasilocally"

$$V^{(+)} := -\frac{v}{r^2}\left[(\overset{\circ}{\Delta} + 2)^2\left(\overset{\circ}{\Delta} - \frac{6m}{r}\right) + \frac{36m^2}{r^2}\left(\overset{\circ}{\Delta} + 2 - \frac{2m}{r}\right)\right]\left(\overset{\circ}{\Delta} + 2 - \frac{6m}{r}\right)^{-2}$$

If we insert the invariant $\mathbf{x}$ in an analogous special form $\mathbf{x} = \exp(i\sigma t)Y_l(\theta,\phi)Z^{(+)}(r)/r$ into equation (4.30) we obtain *Zerilli equation* [2], [4], [5]

$$\left(\frac{d^2}{dr_*^2} + \sigma^2\right)Z^{(+)} = V^{(+)}Z^{(+)}$$

This way we have shown that both equations (Regge-Wheeler and Zerilli) posses gauge invariant formulations and their solutions contain the entire gauge–independent information about the linearized gravitational field on the Schwarzschild background (see also [5]).

Let us notice that $\mathbf{x}$ and $\mathbf{y}$ are scalars on each sphere $S_t(r)$ with respect to the coordinates x^A. For the scalar f on a sphere we can define a "monopole" part mon(f) and a "dipole" part dip(f) as a corresponding component with respect to the spherical harmonics on S^2. Similarly, the "dipole" part of a vector v^A corresponds to the dipole harmonics for the scalars $v^A{}_{||A}$ and $\varepsilon^{AB}v_{A||B}$. Let us denote by $\underline{f}$ the "mono–dipole–free" part of f. According to this decomposition we have

$$\mathbf{x} = \mathrm{mon}(\mathbf{x}) + \underline{\mathbf{x}}$$

$$\mathbf{y} = \mathrm{dip}(\mathbf{y}) + \underline{\mathbf{y}}$$

The dipole part of $\mathbf{x}$ and monopole part of $\mathbf{y}$ are vanishing[3]. The rest of the mono-dipole part of each scalar can be solved explicitly from the equations (4.23)–(4.28) and the solution has the form

$$\mathbf{x} - \underline{\mathbf{x}} = \frac{4\mathbf{m}}{r - 3m}$$

$$\mathbf{y} - \underline{\mathbf{y}} = \frac{12\mathbf{s}}{r^2}$$

From (4.23), (4.27) and the observation that mon$(\mathbf{X}) = $ dip$(\mathbf{Y}) = 0$ we obtain

$$\dot{\mathbf{m}} = \dot{\mathbf{s}} = 0$$

[3] This is included in the definitions (4.21) and (4.19). More precisely, $\mathbf{y}$ is a divergence (see also (B.1)) and $\mathbf{x} = r^2\chi^{AB}{}_{||AB} + (\overset{\circ}{\Delta} + 2)[...]$. Moreover, dip$(\chi^{AB}{}_{||AB}) = 0$ because double-divergence of any traceless tensor is "mono-dipole-free" (see also (B.2) in Appendix B).

Moreover, $\overset{\circ}{\Delta} m = 0$, $(\overset{\circ}{\Delta} + 2)\mathbf{s} = 0$, which simply means that $\mathbf{m}$ is a monopole and $\mathbf{s}$ is a dipole, and they are constant with respect to the coordinates t, r. They correspond to the charges introduced in section 3. More precisely, $\mathbf{m} = p^0$. Moreover, the angular momentum charge (3.3) can be obtained from the relation between spatial constant three-vector in cartesian coordinates and dipole harmonics

$$\mathbf{s} = \frac{s^l z_l}{r}$$

where z_l are cartesian coordinates, $r = \sqrt{\delta^{kl} z_k z_l}$ and s^l is a corresponding three-vector representing angular momentum (see [19]).

5 Stationary solutions

For the axial degree of freedom $\mathbf{y}$ we can rewrite equation (4.26) using a new coordinate $z := \frac{2m}{r}$.

$$4m^2 \ddot{\mathbf{y}} = (1-z)^2 z^4 \frac{\partial^2 \mathbf{y}}{\partial z^2} - (1-z) z^4 \frac{\partial \mathbf{y}}{\partial z} + (1-z) z^2 (\overset{\circ}{\Delta} + 4z)\mathbf{y}$$

This way horizon corresponds to $z = 1$ and spatial infinity to $z = 0$. Let us consider stationary solutions of the above equation which are regular at the spatial infinity (corresponding to $z = 0$).

$$(1-z) z^2 \frac{\partial^2 \mathbf{y}}{\partial z^2} - z^2 \frac{\partial \mathbf{y}}{\partial z} + (\overset{\circ}{\Delta} + 4z)\mathbf{y} = 0 \tag{5.1}$$

Moreover, if we separate the angular variables and include standard asymptotic behaviour at $z = 0$:

$$\mathbf{y} = z^{l+1} Y_l(x^A) u(z)$$

then the equation for the function u is relatively simple

$$(1-z) z u'' + [2l + 1 - (2l+3)z] u' - (l-1)(l+3) u = 0$$

and the solution regular at $z = 0$ is given by the hypergeometric function

$$u = F(l-1, l+3, 2l+2; z)$$

In particular for $l = 1$ the function u is constant, it represents the angular momentum charge solution (3.3), and corresponding $\mathbf{y}$ is finite on the horizon. On the other hand, for $l \geq 2$ we obtain logarithmic divergence of the hypergeometric function F at $z = 1$. More precisely,

$$F(l-1, l+3, 2l+2; z) = z^{-2l-1}[P(z) + \tilde{P}(z) \ln(1-z)]$$

where P and $\tilde{P}$ are polynomials. The solution is not regular at $z = 1$ and it confirms the result of Vishveshwara [3] that the only nontrivial stationary perturbation is given by the axial perturbation with $l = 1$. The interpretation of this solution is given by (3.3). One can check by direct computation for the Kerr metric [10], [11]:

$$\rho^2 := r^2 + a^2 \cos^2\theta \quad \triangle := r^2 - 2mr + a^2$$

$$g_{00} = -1 + \frac{2mr}{\rho^2} \quad g_{0\phi} = -\frac{2mra \sin^2\theta}{\rho^2} \quad g_{rr} = \frac{\rho^2}{\triangle} \quad g_{\theta\theta} = \rho^2$$

$$g_{\phi\phi} = \sin^2\theta \left(r^2 + a^2 + \frac{2mra^2 \sin^2\theta}{\rho^2} \right)$$

that the infinitesimal angular momentum gives the same result. More precisely, the linear part of the Kerr metric $g_{\mu\nu}$ with respect to the parameter a:

$$g_{\mu\nu} \mathrm{d}x^\mu \mathrm{d}x^\nu = \eta_{\mu\nu} \mathrm{d}x^\mu \mathrm{d}x^\nu - 4\frac{ma}{r} \sin^2\theta \mathrm{d}t \mathrm{d}\phi + O(a^2)$$

gives only the one non-vanishing component of the linearized Kerr metric on the Schwarzschild background:

$$h_{0\phi} = -\frac{2ma}{r}\sin^2\theta$$

From the definition (4.19) and the equation (4.9) we can calculate the invariant $\mathbf{y} = 12ma\cos\theta/r^2$. It is easy to compare the result with (3.3) and it gives angular momentum charge $s^z = ma$.

Can we consider monopole solution in axial invariant? The definition (4.19) in terms of the initial data does not allow nontrivial monopole part of the axial invariant. However, we can consider such situation if we admit singular metric. Formally this case corresponds to the infinitesimal Taub-NUT solution. The Taub-Nut metric [12], [13]:

$$g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = (r^2+l^2)(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2) + \tilde{v}^{-1}\mathrm{d}r^2 - \tilde{v}(\mathrm{d}t + 2l\cos\theta \mathrm{d}\phi)^2 =$$

$$= \eta_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu - 4vl\cos\theta \mathrm{d}t\mathrm{d}\phi + O(l^2)\,; \quad \tilde{v} := \frac{r^2-2mr-l^2}{r^2+l^2}$$

gives the linearized metric $h_{0\phi} = -2lv\cos\theta$ and the monopole axial invariant $\mathbf{y} = -\frac{4l}{r}(1-\frac{3m}{r})$. We should stress that this is only formal calculation because the tensor $h_{0\phi} = -2lv\cos\theta$ is not well defined along the $z-$axis and is excluded as a global solution. The monopole charge in $\mathbf{y}$ plays a role of the topological obstruction for the existence of the global metric. It is similar to the magnetic monopole in electrodynamics (see also [19]).

For polar degree of freedom $\mathbf{x}$ from equation (4.30) we get

$$4m^2\ddot{\mathbf{x}} = (1-z)^2 z^4 \frac{\partial^2 \mathbf{x}}{\partial z^2} - (1-z)z^4\frac{\partial \mathbf{x}}{\partial z} + (1-z)z^2\left[\frac{1}{3} + \frac{2}{3}(\overset{\circ}{\Delta}-1)(\overset{\circ}{\Delta}+2)(\overset{\circ}{\Delta}+2-3z)^{-2}\right](\overset{\circ}{\Delta}+2)\mathbf{x}$$

and the corresponding stationary equation has the following form

$$(1-z)z^2\frac{\partial^2\mathbf{x}}{\partial z^2} - z^2\frac{\partial \mathbf{x}}{\partial z} + \left[\frac{1}{3} + \frac{2}{3}(\overset{\circ}{\Delta}-1)(\overset{\circ}{\Delta}+2)(\overset{\circ}{\Delta}+2-3z)^{-2}\right](\overset{\circ}{\Delta}+2)\mathbf{x} = 0 \tag{5.2}$$

For $l = 0$ the solution $\mathbf{x} = \frac{z}{2-3z}$ is related to (3.5) and corresponds to the mass charge. More precisely, if we put in the metric (2.7) $m + \delta m$ instead of parameter m and take the linear part in δm we get

$$\eta_{\mu\nu}(m+\delta m)\mathrm{d}x^\mu \mathrm{d}x^\nu = \eta_{\mu\nu}(m)\mathrm{d}x^\mu \mathrm{d}x^\nu + \frac{2\delta m}{r}\mathrm{d}t^2 + \frac{2\delta m}{v^2 r}\mathrm{d}r^2 + O(\delta m^2)$$

and the invariant $\mathbf{x} = 2\mathcal{B}h^{33} = \frac{4\delta m}{r}\left(1-\frac{3m}{r}\right)^{-1}$. If we compare with mass charge (3.5) we obtain $p^0 = \delta m$ (see also [1]).

For $l \geq 2$ we have the following transformation law:

$$\overset{\circ}{\Delta}(\overset{\circ}{\Delta}+2)\mathbf{x} = -6z^2(1-z)\frac{\partial \mathbf{y}}{\partial z} + \left[\overset{\circ}{\Delta}(\overset{\circ}{\Delta}+2) + 6z(1-z) - 18z^2(1-z)(\overset{\circ}{\Delta}+2-3z)^{-1}\right]\mathbf{y}$$

which moves the solution of (5.1) into solutions of (5.2). In particular, it is clear that for $l \geq 2$ the stationary solutions of the equation (5.2) have the same logarithmic divergence on the horizon as the solutions of the equation (5.1). The explicit stationary solutions (in a specific gauge) were also given by Zerilli in [2] and here we present a gauge invariant confirmation of his result.

Remark Although $l = 1$ is excluded in the definition of $\mathbf{x}$, we can consider another variable

$$(\overset{\circ}{\Delta}+2)^{-1}\mathbf{x} := (\overset{\circ}{\Delta}+2)^{-1}r^2\chi^{AB}{}_{||AB} - \frac{1}{2}H + \left(\overset{\circ}{\Delta}+2-\frac{6m}{r}\right)^{-1}[2h^{33} + 2rh^{3A}{}_{||A} - rvH_{,3}]$$

which is no longer gauge-invariant in its dipole part (see also (A.8) in Appendix A). More precisely, it transforms with respect to the gauge transformation as follows

$$\mathrm{dip}\left(\frac{1}{2}H + \frac{r}{6m}vQ\right) = \mathrm{dip}(-(\overset{\circ}{\Delta}+2)^{-1}\mathbf{x}) \longrightarrow \mathrm{dip}(-(\overset{\circ}{\Delta}+2)^{-1}\mathbf{x}) + \mathrm{dip}(\xi^A{}_{||A})$$

where Q is defined in Appendix A by (A.2). Formally, the dipole solution $(\overset{\circ}{\Delta}+2)^{-1}\mathbf{x} = \ln(1-z)Y_1(x^A)$ fulfills the same equation (5.2), and it corresponds to the Regge-Wheeler infinitesimal translation as a gauge transformation[4] (eq. 32 in [1]):

$$\xi^3 = \cos\theta\,; \quad \xi^\theta = \frac{\sin\theta}{2m}\ln\left(1-\frac{2m}{r}\right)\,; \quad \xi^A{}_{||A} = \frac{\cos\theta}{m}\ln\left(1-\frac{2m}{r}\right)$$

This way the stationary polar dipole solution corresponds to the infinitesimal translation gauge and it is also logarithmically divergent on the horizon.

6 The symplectic form and its reduction

In this section we show the relation between the symplectic structure and the invariants introduced in the section 4. Let (P^{kl}, h_{kl}) be the Cauchy data on a hypersurface Σ. Let us consider the symplectic structure

$$\Omega := \int_\Sigma \delta P^{kl}\wedge\delta h_{kl}$$

It is invariant with respect to the spatial gauge transformation which is fixed on the boundary $(\delta\xi^k\big|_{\partial\Sigma}=0)$:

$$\int_\Sigma \delta P^{kl}\wedge\delta h_{kl} \quad \xrightarrow{\xi^k} \quad \int_\Sigma \delta P^{kl}\wedge\delta h_{kl} + 2\int_{\partial\Sigma}\delta P^3{}_l\wedge\delta\xi^l$$

Moreover, it is invariant with respect to the temporal gauge if we assume that ξ^0 and its normal derivative[5] are fixed on the boundary:

$$\int_\Sigma \delta P^{kl}\wedge\delta h_{kl} \quad \xrightarrow{\xi^0} \quad \int_\Sigma \delta P^{kl}\wedge\delta h_{kl} + \\ + \int_{\partial\Sigma}\sqrt{\eta}\left[\delta(N\xi^0)_{|k}\wedge\delta h^{3k} - \delta(N\xi^0)^{|3}\wedge\delta h + N\delta\xi^0\wedge\delta(h^{|3}-h^{3l}{}_{|l})\right]$$

Roughly speaking, the symplectic structure is invariant with respect to the gauge modulo boundary terms. The quadratic form $\int_\Sigma \delta P^{kl}\wedge\delta h_{kl}$ can be decomposed into monopole part, dipole part and the remainder in a natural way.

From the considerations given in the Appendix B (formulae (B.5) and (B.13)) we can easily see that the "radiation" part

$$\underline{\Omega} = \int_\Sigma \delta\underline{P}^{kl}\wedge\delta\underline{h}_{kl} \sim \int_\Sigma \delta\underline{\mathbf{X}}\wedge\overset{\circ}{\Delta}{}^{-1}(\overset{\circ}{\Delta}+2)^{-1}\delta\underline{\mathbf{x}} + \delta\underline{\mathbf{Y}}\wedge\overset{\circ}{\Delta}{}^{-1}(\overset{\circ}{\Delta}+2)^{-1}\delta\underline{\mathbf{y}}$$

where symbol "$\sim$" denotes equality modulo boundary term. Moreover, the "mono-dipole" part has the form (see (B.9) and (B.4) respectively)

$$\mathrm{mon}\left(\int_\Sigma \delta P^{kl}\wedge\delta h_{kl}\right) = \int_\Sigma \frac{1}{2}\delta P_{33}\wedge\delta\mathcal{B}^{-1}\mathrm{mon}(\mathbf{x}) + \frac{1}{2}\int_{\partial\Sigma} r\delta P^3{}_3\wedge\delta\mathrm{mon}(H) \tag{6.1}$$

[4] If we assume that the translation corresponds to the $\xi^3=\cos\theta$ then the component ξ^θ is uniquely defined as the polar gauge transformation preserving the gauge condition $h_3{}^A{}_{||A}=0$ which has been used by Regge-Wheeler [1] and Vishveshwara [3].

[5] You may "correct" the symplectic form by a boundary term in such a way that the result is gauge invariant for ξ^μ vanishing on the $\partial\Sigma$ without an extra assumption about derivatives (see [20]).

$$\mathrm{dip}(\int_\Sigma \delta P^{kl}\wedge\delta h_{kl}) \sim -\int_\Sigma \Lambda\delta\mathrm{dip}(\mathbf{y})\wedge \overset{\circ}{\Delta}{}^{-1}\delta(h_{3A||B}\varepsilon^{AB}) \tag{6.2}$$

The monopole part of the polar invariant $\mathrm{mon}(\mathbf{x})$ and the dipole part of the axial one $\mathrm{dip}(\mathbf{y})$ represent mass and angular momentum respectively, and they are supposed to be fixed. These quantities are analogous to the electric charge in electrodynamics. If we assume that there is no matter inside volume Σ then both of them are fixed by the constraints, provided that they are controlled at $r = 2m$ as a boundary condition on the horizon. Let us assume that $\delta\mathrm{mon}(\mathbf{x})|_{S(r=2m)} = 0$, $\delta\mathrm{dip}(\mathbf{y})|_{S(r=2m)} = 0$ then the mono-dipole part (6.1) and (6.2) vanish and the symplectic structure reduces to the "mono-dipole-free" invariants

$$\int_\Sigma \delta P^{kl}\wedge\delta h_{kl} \sim \int_\Sigma \delta\underline{\mathbf{X}}\wedge \overset{\circ}{\Delta}{}^{-1}(\overset{\circ}{\Delta}+2)^{-1}\delta\underline{\mathbf{x}} + \delta\underline{\mathbf{Y}}\wedge \overset{\circ}{\Delta}{}^{-1}(\overset{\circ}{\Delta}+2)^{-1}\delta\underline{\mathbf{y}} \tag{6.3}$$

This way we obtain $\underline{\mathbf{X}}, \underline{\mathbf{x}}, \underline{\mathbf{Y}}, \underline{\mathbf{y}}$ as the quasi-local canonical variables describing reduced unconstrained initial data on Σ.

Remark One can ask the question when (6.3) is a strict equality not only modulo boundary term. The symplectic 2-form Ω reduces to the mono-dipole-free invariants if we assume the following boundary conditions which fix the gauge freedom on the boundary:

$$\delta h_{AB}|_{\partial\Sigma} = 0 \quad \delta(2h^3_3 + 2rh_3{}^A{}_{||A} - rH_{,3})|_{\partial\Sigma} = 0$$

$$\delta(P^{3A||B}\varepsilon_{AB})\Big|_{\partial\Sigma} = 0 \quad \delta(P^3_3 + 2r\,\overset{\circ}{\Delta}{}^{-1} P^{3A}{}_{||A})\Big|_{\partial\Sigma} = 0 \tag{6.4}$$

In Appendix A we analyze the possibility when $\underline{H}$, $\underline{Q}$, $\underline{\Pi}$ and $\chi^{AC}{}_{||CB}\varepsilon_A{}^B$ are precisely the gauge conditions and then $\chi^{AB}{}_{||AB}$ corresponds to the control of $\underline{\mathbf{x}}$. Roughly speaking, the "control mode" given by (6.4) contains four boundary conditions related to the gauge freedom plus two boundary conditions for the unconstrained degrees of freedom which we propose to call Dirichlet boundary data.

If we introduce quasilocal coordinates:

$$q^1 := \overset{\circ}{\Delta}{}^{-1/2}(\overset{\circ}{\Delta}+2)^{-1/2}\underline{\mathbf{x}}\,; \quad p_1 := \overset{\circ}{\Delta}{}^{-1/2}(\overset{\circ}{\Delta}+2)^{-1/2}\underline{\mathbf{X}}$$

$$q^2 := \overset{\circ}{\Delta}{}^{-1/2}(\overset{\circ}{\Delta}+2)^{-1/2}\underline{\mathbf{y}}\,; \quad p_2 := \overset{\circ}{\Delta}{}^{-1/2}(\overset{\circ}{\Delta}+2)^{-1/2}\underline{\mathbf{Y}}$$

we can rewrite the reduced symplectic structure (6.3) in the canonical form

$$\Omega = \int_\Sigma \delta\underline{\mathbf{X}}\wedge \overset{\circ}{\Delta}{}^{-1}(\overset{\circ}{\Delta}+2)^{-1}\delta\underline{\mathbf{x}} + \delta\underline{\mathbf{Y}}\wedge \overset{\circ}{\Delta}{}^{-1}(\overset{\circ}{\Delta}+2)^{-1}\delta\underline{\mathbf{y}} = \sum_{n=1}^{2}\int_\Sigma \delta p_n\wedge\delta q^n$$

Unconstrained initial data for the full nonlinear theory (which is similar to the considerations in this article) has been proposed in [18]. Moreover, the concept of quasilocality appeared in [17] and has been developed in [20]. The boundary data possesses its counterpart in the full nonlinear theory (see proposition in [20]) and it will be discussed elsewhere.

7 Energy and angular momentum of the gravitational waves

The reduction of the symplectic form (6.3) allows to formulate the hamiltonian relation in terms of the reduced canonical variables

$$\int_\Sigma \dot{\mathbf{x}}\,\overset{\circ}{\Delta}{}^{-1}(\overset{\circ}{\Delta}+2)^{-1}\delta\mathbf{X} - \dot{\mathbf{X}}\,\overset{\circ}{\Delta}{}^{-1}(\overset{\circ}{\Delta}+2)^{-1}\delta\mathbf{x} +$$

$$+\int_\Sigma \dot{\mathbf{y}}\,\overset{\circ}{\Delta}{}^{-1}(\overset{\circ}{\Delta}+2)^{-1}\delta\mathbf{Y} - \dot{\mathbf{Y}}\,\overset{\circ}{\Delta}{}^{-1}(\overset{\circ}{\Delta}+2)^{-1}\delta\mathbf{y} = 16\pi\delta\mathcal{H} +$$

$$+\int_{\partial\Sigma}\frac{\Lambda}{r}v(r\mathbf{x})_{,3}\ \mathring{\Delta}^{-1}(\ \mathring{\Delta}+2)^{-1}\delta\mathbf{x}+\frac{\Lambda}{r}v(r\mathbf{y})_{,3}\ \mathring{\Delta}^{-1}(\ \mathring{\Delta}+2)^{-1}\delta\mathbf{y} \tag{7.1}$$

where

$$16\pi\mathcal{H}:=\frac{1}{2}\int_{\Sigma}\frac{v}{\Lambda}\mathbf{X}\ \mathring{\Delta}^{-1}(\ \mathring{\Delta}+2)^{-1}\mathbf{X}+\frac{v}{\Lambda}\mathbf{Y}\ \mathring{\Delta}^{-1}(\ \mathring{\Delta}+2)^{-1}\mathbf{Y}+$$

$$+\frac{1}{2}\int_{\Sigma}\frac{\Lambda}{r^2}\left[v(r\mathbf{x})_{,3}\ \mathring{\Delta}^{-1}(\ \mathring{\Delta}+2)^{-1}(r\mathbf{x})_{,3}+\mathbf{x}\frac{r^2}{v}V^{(+)}\ \mathring{\Delta}^{-1}(\ \mathring{\Delta}+2)^{-1}\mathbf{x}\right]+$$

$$+\frac{1}{2}\int_{\Sigma}\frac{\Lambda}{r^2}\left[v(r\mathbf{y})_{,3}\ \mathring{\Delta}^{-1}(\ \mathring{\Delta}+2)^{-1}(r\mathbf{y})_{,3}+\mathbf{y}\frac{r^2}{v}V^{(-)}\ \mathring{\Delta}^{-1}(\ \mathring{\Delta}+2)^{-1}\mathbf{y}\right] \tag{7.2}$$

(see also eq. 4.19 and 5.34 in [5]).
Similarly for angular momentum we propose the following expression

$$16\pi J(Z)=\int_{\Sigma}\mathbf{X}\ \mathring{\Delta}^{-1}(\ \mathring{\Delta}+2)^{-1}Z^A\partial_A\mathbf{x}+\mathbf{Y}\ \mathring{\Delta}^{-1}(\ \mathring{\Delta}+2)^{-1}Z^A\partial_A\mathbf{y} \tag{7.3}$$

where Z is a Killing field (3.1). In particular for $Z=\partial/\partial\phi$ the $z-$component of the angular momentum takes the form:

$$16\pi J_z=\int_{\Sigma}\mathbf{X}\ \mathring{\Delta}^{-1}(\ \mathring{\Delta}+2)^{-1}\mathbf{x}_{,\phi}+\mathbf{Y}\ \mathring{\Delta}^{-1}(\ \mathring{\Delta}+2)^{-1}\mathbf{y}_{,\phi} \tag{7.4}$$

The conservation laws for the energy and angular momentum

$$\partial_0\mathcal{H}=0$$

$$\partial_0 J(Z)=0$$

are fulfilled if we assume appropriate boundary conditions on the horizon and at the spatial infinity. The natural choice of those conditions is to assume that $\underline{\mathbf{x}}$ and $\underline{\mathbf{y}}$ are vanishing on the boundary $\partial\Sigma$.

After separation of the angular variables in (7.2) we obtain a hamiltonian which has been used for the energy method by Wald [6] (see also in [4] eq. 386) and it confirms stability of the Schwarzschild metric (see also [5]). Here we have shown how to combine an energy of different multipoles together.

7.1 Regular initial data

If we assume that the invariants $\mathbf{x}$, $\mathbf{y}$ are vanishing on the horizon then we get a nice hamiltonian system outside of the horizon. We can also assume that $\underline{\mathbf{x}}$ and $\underline{\mathbf{y}}$ are fixed and finite on the horizon. More precisely, we propose the following initial-boundary data:
1. mon($\mathbf{x}$)=0, because of the singularity at $r=3m$, moreover this charge indicates that we have chosen wrong parameter m in the background.
2. dip($\mathbf{y}$) – weak internal angular momentum.
3. The radiation data ($\underline{\mathbf{x}}$, $\underline{\mathbf{X}}$, $\underline{\mathbf{y}}$, $\underline{\mathbf{Y}}$) has to be finite on the horizon, moreover $\underline{\mathbf{x}}$, $\underline{\mathbf{y}}$ should be controlled as a Dirichlet boundary condition. We assume also that $h_{\mu\nu}=O(1/r)$ at spatial infinity. The center of mass contained in dip[$(\ \mathring{\Delta}+2)^{-1}\mathbf{x}$] and linear momentum in dip(Π) can be always "gauged out", see Appendix A.1, and those gauge transformations correspond to the infinitesimal translation and boost respectively. Performing those gauge transformations we pass to the "center mass rest frame".
4. The asymptotics of the invariants $\underline{\mathbf{x}}$, $\underline{\mathbf{y}}$ at spatial infinity should guarantee finite hamiltonian and this can be achieved for standard asymptotics $h_{\mu\nu}=O(1/r)$. Unfortunately, the standard asymptotics $\frac{1}{r}$ does not guarantee finite angular momentum. More precisely, standard asymptotics at spatial infinity $\underline{\mathbf{x}},\underline{\mathbf{y}}=O(1/r)$, $\dot{\underline{\mathbf{x}}},\dot{\underline{\mathbf{y}}}=O(1/r^2)$ gives logarithmically divergent integral in (7.4). The proposition by Christodolou and Kleinerman [14], so-called S.A.F. condition, fits perfectly if we adopt it to the linearized theory. We propose the following asymptotics at spatial infinity

$$\mathbf{x},\mathbf{y}=O(r^{-3/2})\,,\quad \dot{\mathbf{x}},\dot{\mathbf{y}}=O(r^{-5/2})$$

Remark Christodolou and Kleinerman assume that the full ADM data has better asymptotics (except conformal factor in the riemannian metric). Let us notice that if we assume that $P^{kl} = O(r^{-1/2})$ and $h_{kl} = O(r^{-3/2})$ then all boundary terms in the symplectic form (analyzed in Appendix B) are vanishing like $\frac{1}{r}$ at spatial infinity.

8 Conclusions

We have shown a natural functionals which represent energy and angular momentum of the weak gravitational radiation on a Schwarzschild background. Particularly, we do not have to separate angular variables and the result is gauge-invariant. Moreover, the equations of motion for the gauge-invariant degrees of freedom correspond to the well-known Regge-Wheeler and Zerilli results. We give also a complete (together with the interpretation) analysis of the stationary solutions.

After separation of the angular variables we should notice that the results presented in sections 4, 6 and 7 are very close to the approach presented by Moncrief in [5]. However, the invariants used by Moncrief in polar part are different and non-local.

A Reduction of the initial data to the invariants

It is convenient to introduce the following variables:

$$\Pi := 2rP^{3A}{}_{||A} + \overset{\circ}{\Delta} P^3{}_3 \tag{A.1}$$

$$Q := 2h^3_3 + 2rh_3{}^A{}_{||A} - rH_{,3} \tag{A.2}$$

The scalar constraint (4.17) takes the form

$$\frac{\sqrt{v}}{r}(r^2\sqrt{v}Q)_{,3} + r^2\chi^{AB}{}_{||AB} - \frac{1}{2}(\overset{\circ}{\Delta} + 2)H - \left(\overset{\circ}{\Delta} + 2 - \frac{6m}{r}\right)h^3_3 = 0 \tag{A.3}$$

From vector constraint (4.15-4.16) we get

$$r\sqrt{v}(\sqrt{v}\Pi)_{,3} + \left(\overset{\circ}{\Delta} + 2 - \frac{6m}{r}\right)rP^{3A}{}_{||A} + 2vr^2S^{AB}{}_{||AB} = 0 \tag{A.4}$$

A.1 Dipole polar part of the initial data

In this subsection we consider only dipole parts of the variables and we denote by the same letter their dipole parts as the full objects in the rest of the paper.

1. From vector constraint (A.4), (4.15) and the definition (A.1) we have

$$rP^{3A}{}_{||A} = \frac{r^2\sqrt{v}}{6m}(\sqrt{v}\Pi)_{,3} \tag{A.5}$$

$$P^3{}_3 = \frac{r^2\sqrt{v}}{6m}(\sqrt{v}\Pi)_{,3} - \frac{1}{2}\Pi \tag{A.6}$$

$$S = \left[\frac{r^2\sqrt{v}}{6m}(\sqrt{v}\Pi)_{,3}\right]_{,3} \tag{A.7}$$

This means that Π contains the full information about dipole polar part of P^{kl}.

2. From scalar constraint (A.3) we get

$$h^3_3 = -\frac{\sqrt{v}}{6m}(r^2\sqrt{v}Q)_{,3}$$

Moreover, from the definition (A.2) we have

$$2rh_3{}^A{}_{||A} = Q + \frac{\sqrt{v}}{3m}(r^2\sqrt{v}Q)_{,3} + rH_{,3}$$

This way we can see that Q and H contain the full information about dipole polar part of the linearized metric h_{kl}.

3. The polar dipole gauge can be described by the following transformations:

$$\begin{aligned}
&-\frac{r^3}{12m\Lambda}\Pi \longrightarrow -\frac{r^3}{12m\Lambda}\Pi + \xi^0 \\
&-\frac{rv}{6m}Q \longrightarrow -\frac{rv}{6m}Q + \xi^3 \\
&\frac{1}{2}H + \frac{rv}{6m}Q \longrightarrow \frac{1}{2}H + \frac{rv}{6m}Q + \xi^A{}_{||A}
\end{aligned} \tag{A.8}$$

and they show that we can always perform quasilocal gauge transformation in such a way that Π, Q and H are vanishing.

Moreover, from (4.8–4.12) we can check the evolution equations:

$$\begin{aligned}
&-h^0_0 = \frac{r^3}{6m\Lambda}\dot{\Pi} + \frac{1}{6}Q \\
&h_{03} = \frac{r^2 v}{12m\Lambda}(r\Pi)_{,3} - \frac{r^2}{12m}\dot{Q} \\
&h_0{}^A{}_{||A} = \frac{1}{2}\dot{H} + \frac{rv}{6m}\dot{Q} - \frac{rv}{6m\Lambda}\Pi
\end{aligned}$$

Finally we have shown that quasilocal gauge $\Pi = Q = H = 0$ gives vanishing dipole polar part of the full metric $h_{\mu\nu}$.

A.2 Radiation polar part of the initial data in Regge-Wheeler gauge

The special form of the metric $h_{\mu\nu}$ proposed in [1] will be called Regge-Wheeler gauge[6]. In polar part it gives the following gauge conditions:

$$\begin{aligned}
&\chi^{AB}{}_{||AB} = h_{0A}{}^{||A} = h_{3A}{}^{||A} = 0\,; \quad r^2\chi^{AB}{}_{||AB} \longrightarrow r^2\chi^{AB}{}_{||AB} + (\overset{\circ}{\Delta} + 2)\xi^A{}_{||A} \\
&r^2 h_{0A}{}^{||A} \longrightarrow r^2 h_{0A}{}^{||A} + \overset{\circ}{\Delta}\xi_0 + r^2\dot{\xi}^A{}_{||A}\,; \quad r^2 h_{3A}{}^{||A} \longrightarrow r^2 h_{3A}{}^{||A} + \overset{\circ}{\Delta}\xi_3 + r^2(\xi^A{}_{||A})_{,3}
\end{aligned} \tag{A.9}$$

which allow to reconstruct polar part of the metric $h_{\mu\nu}$ as follows:

1. Equation

$$2v\Lambda^{-1}S^{AB}{}_{||AB} = \dot{\chi}^{AB}{}_{||AB} - (\overset{\circ}{\Delta} + 2)h_{0A}{}^{||A}$$

follows directly from (4.10) and gives $S^{AB}{}_{||AB} = 0$. The variable Π (defined by (A.1)) contains the same information as $\underline{\mathbf{X}} = 2r^2S^{AB}{}_{||AB} + \mathcal{B}\underline{\Pi}$. From (A.4) we reconstruct $\underline{P}^{3A}{}_{||A}$, and (A.1) gives $\underline{P}^3{}_3$. Finally, we reconstruct $\underline{S}$ from vector constraint (4.15). This way we have obtained four polar components of the ADM momentum $\underline{P}^{kl}$, namely $S^{AB}{}_{||AB}$, $\underline{P}^{3A}{}_{||A}$, $\underline{P}^3{}_3$ and $\underline{S}$. The remaining two axial components are analyzed in subsection A.4.

[6] Regge-Wheeler impose maximal number of gauge conditions: 1 axial gauge (A.10) and 3 polar gauge conditions (A.9) in radiation part. They do not discuss mono-dipole part except some stationary solutions. Chandrasekhar in his book [4] does not impose axial gauge. Vishveshwara uses the same formalism as Regge-Wheeler plus axial gauge (A.11) for the dipole and moreover in polar part he assumes dip$(h^0_0 + h^3_3) = 0$. The monopole part is not discussed in the literature.

2. The spatial metric we reconstruct from the scalar constraint (A.3) together with (A.2) and the observation that $\mathcal{B}Q = \mathbf{x} + \frac{1}{2}(\overset{\circ}{\Delta} + 2)H$. More precisely, we get a system of two equations for two missing components $\underline{H}$, $\underline{h}^3{}_3$ of the spatial metric (we need boundary data to solve them!).

3. The lapse $\underline{h}^0_0$ we get from $\dot{S}^{AB}{}_{||AB}$ which is given by (4.13). And finally the missing component $\underline{h}_{03}$ of the shift we can calculate from the following equation

$$2r^2\Lambda^{-1}P^{3A}{}_{||A} + \overset{\circ}{\Delta} h_{03} = r^2\dot{h}_{3A||A} - r^2(h_{0A}{}^{||A})_{,3}$$

which may be easily checked from (4.9). The axial components of the spatial metric and axial part of the shift vector we analyze in subsection A.4.

The above analysis together with the results in subsection A.4 show how to reconstruct the full initial data $(\underline{P}^{kl}, \underline{h}_{kl})$ together with lapse $\underline{h}^0_0$ and shift $\underline{h}_{0k}$ in the Regge-Wheeler gauge (A.9) and (A.10) from the reduced initial data ($\underline{\mathbf{x}}$, $\underline{\mathbf{y}}$, $\underline{\mathbf{X}}$, $\underline{\mathbf{Y}}$).

A.3 Radiation polar part of the initial data in quasilocal gauge

We do not like to impose any conditions directly on lapse and shift. It is more elegant to impose gauge conditions on the initial data only. The "time conservation laws" of the gauge conditions obtained from equations of motion give lapse and shift indirectly. For this purpose we propose the following quasilocal gauge conditions

$$\underline{Q} = \underline{H} = \underline{\Pi} = 0$$

which allow to reconstruct radiation polar part of the four-metric in a quasilocal way.

$$rvQ \longrightarrow rvQ + 2\left(\overset{\circ}{\Delta} + 2 - \frac{6m}{r}\right)\xi^3$$

$$\frac{r^2}{\Lambda}\Pi \longrightarrow \frac{r^2}{\Lambda}\Pi - \frac{1}{2}\overset{\circ}{\Delta}\left(\overset{\circ}{\Delta} + 2 - \frac{6m}{r}\right)\xi^0$$

$$vQ - \frac{1}{2}\left(\overset{\circ}{\Delta} + 2 - \frac{6m}{r}\right)H \longrightarrow vQ - \frac{1}{2}\left(\overset{\circ}{\Delta} + 2 - \frac{6m}{r}\right)H - \left(\overset{\circ}{\Delta} + 2 - \frac{6m}{r}\right)\xi^A{}_{||A}$$

Moreover, each component of the metric $\underline{h}_{\mu\nu}$ depends quasilocally on the invariants. This can be shown as follows:

1. From (A.2) and (A.3) we obtain metric components

$$r^2\chi^{AB}{}_{||AB} = \underline{\mathbf{x}}\,, \quad \underline{H} = 0$$

$$\underline{h}^3_3 = -r\underline{h}_{3A}{}^{||A} = \left(\overset{\circ}{\Delta} + 2 - \frac{6m}{r}\right)^{-1}\underline{\mathbf{x}}$$

2. From (A.1), (A.4) and (4.15) we get the ADM momentum

$$2r^2 S^{AB}_{||AB} = \underline{\mathbf{X}}\,, \quad \underline{S} = \left[2vr\,\overset{\circ}{\Delta}{}^{-1}\left(\overset{\circ}{\Delta} + 2 - \frac{6m}{r}\right)^{-1}\underline{\mathbf{X}}\right]_{,3} - \overset{\circ}{\Delta}{}^{-1}\underline{\mathbf{X}}$$

$$\overset{\circ}{\Delta}\underline{P}^3_3 = -2r\underline{P}^{3A}{}_{||A} = 2v\left(\overset{\circ}{\Delta} + 2 - \frac{6m}{r}\right)^{-1}\underline{\mathbf{X}}$$

3. Moreover,

$$r\Pi - \Lambda r\dot{Q} = \left(\overset{\circ}{\Delta} + 2 - \frac{6m}{r}\right)(rP^3{}_3 - 2\Lambda h_{03})$$

and

$$v\dot{Q} - \frac{1}{2}\left(\overset{\circ}{\Delta} + 2 - \frac{6m}{r}\right)\dot{H} = \Lambda^{-1}v\Pi - \left(\overset{\circ}{\Delta} + 2 - \frac{6m}{r}\right){h_0}^A{}_{||A}$$

give the following components of the shift vector

$$\underline{h}_{0A}{}^{||A} = 0\,, \quad \underline{h}_{03} = \frac{v}{\Lambda}\,\overset{\circ}{\Delta}{}^{-1}\left(\overset{\circ}{\Delta} + 2 - \frac{6m}{r}\right)^{-1}\underline{\mathbf{X}}$$

and finally the equation for $\dot{\Pi}$ gives the lapse function

$$\underline{h}^0{}_0 = \overset{\circ}{\Delta}{}^{-1}\left(\overset{\circ}{\Delta} + 2 - \frac{6m}{r}\right)^{-1}\left(\overset{\circ}{\Delta}\underline{\mathbf{x}} - 2v\mathcal{B}\underline{\mathbf{x}} - 2rv\underline{\mathbf{x}}_{,3}\right)$$

The above quasilocal formulae for the radiation polar part of the metric $h_{\mu\nu}$ allow to check the equations of motion (4.27) and (4.28) by inspection from the (4.8-4.13).

Why we prefer the quasilocal gauge?
Because the initial data with compact support becomes more clear. The full (constrained) initial data with compact support gives the reduced initial data with compact support and the opposite is true only in quasilocal gauge. The data with compact support allows to avoid the boundary value problems.

A.4 Monopole part of the initial data

The monopole part of the data seems to be not analyzed in the literature. We propose here the complete analysis of this simple "gap". Let us prolongate the gauge condition $H = 0$ which fixes the radial coordinate and for the monopole part let us assume

$$\mathrm{mon}(H) = 0$$

This way $\mathrm{mon}(\mathbf{x}) = 2v\mathcal{B}\mathrm{mon}(h^3{}_3)$ and the monopole part of the metric takes the form

$$\mathrm{mon}(H) = 0\,; \quad \mathrm{mon}(h^{33}) = \frac{1}{2}\mathcal{B}^{-1}\mathrm{mon}(\mathbf{x}) = 2\frac{p^0}{r}$$

For time coordinate we propose the following gauge condition

$$\mathrm{mon}(P^3{}_3) = 0$$

which is no longer quasilocal, it needs the boundary data at spatial infinity for the parabolic equation obtained from (4.4). The similar situation (nonlocal reconstruction) we encounter during analysis of the dipole axial part (see the next subsection).

From the vector constraint (4.15) and $\mathrm{mon}(P^3{}_3) = 0$ we obtain

$$\mathrm{mon}(S) = 0$$

This way the whole ADM momentum is trivial in its monopole part.

The trace of (4.10):

$$\frac{1}{2}\dot{H} = \Lambda^{-1}P^{33} + h_{0A}^{||A} + \frac{2v}{r}h_{03}$$

gives $\mathrm{mon}(h_{03}) = 0$, and finally the lapse we get from monopole part of (4.11)

$$\mathrm{mon}(2\frac{r^2}{\Lambda}\dot{P}^3{}_3) = \mathrm{mon}(-2rvh^0_{0,3} + 2vh^3_3)$$

If we assume that h^0_0 vanishes at spatial infinity we obtain

$$\mathrm{mon}(h^0_0) = \frac{p^0}{m}\ln v$$

The infinity which we encounter in the monopole part of the invariant $\mathbf{x}$ at $r = 3m$ suggests that we have to assume $p^0 = 0$ and this way we exclude the possibility of manipulation with the mass parameter m in the background metric.

A.5 Axial part of the initial data

1. The ADM momentum components $P^{3A||B}\varepsilon_{AB}$ and $S^{AC}{}_{CB}\varepsilon_A{}^B$ we get from $\mathbf{y}$ and axial part of the vector constraint (4.18).
2. In radiation part ($l \geq 1$) we impose gauge condition

$$\chi^{AC}{}_{||CB}\varepsilon_A{}^B = 0\,; \quad r^2\chi^{AC}{}_{||CB}\varepsilon_A{}^B \longrightarrow r^2\chi^{AC}{}_{||CB}\varepsilon_A{}^B + (\overset{\circ}{\Delta} + 2)\xi_{A||B}\varepsilon^{AB} \tag{A.10}$$

which fixes $\xi_{A||B}\varepsilon^{AB}$ quasilocally. The component $h_{3A||B}\varepsilon^{AB}$ is obviously contained in $\mathbf{Y}$. Moreover, from (4.10) we have

$$r^2\dot{\chi}^{AC}{}_{||CB}\varepsilon_A{}^B = 2v\frac{r^2}{\Lambda}S^{AC}{}_{||CB}\varepsilon_A{}^B + (\overset{\circ}{\Delta} + 2)h_{0A||B}\varepsilon^{AB}$$

which gives $h_{0A||B}\varepsilon^{AB} = 0$. This way we have shown how to reconstruct axial part of the metric $\underline{h}_{\mu\nu}$ from the invariants $(\underline{\mathbf{y}}, \underline{\mathbf{Y}})$.
3. The dipole part of the metric ($l = 1$) can be fixed by the gauge condition

$$\mathrm{dip}(h_{3A||B}\varepsilon^{AB}) = 0\,; \quad \mathrm{dip}(h_{3A||B}\varepsilon^{AB}) \longrightarrow \mathrm{dip}(h_{3A||B}\varepsilon^{AB}) + \left[\mathrm{dip}(\xi_{A||B}\varepsilon^{AB})\right]_{,3} \tag{A.11}$$

which gives "parabolic" equation for the angular gauge transformation $\mathrm{dip}(\xi_{A||B}\varepsilon^{AB})$ with the boundary value at spatial infinity. Moreover, stationary solution in $\mathrm{dip}(\mathbf{y})$ appears in $h_{0A||B}\varepsilon^{AB}$. More precisely, from "time conservation law of gauge condition"

$$0 = r^2\mathrm{dip}(\dot{h}_{3A||B}\varepsilon^{AB}) = r^2\mathrm{dip}(h_{0A||B}\varepsilon^{AB})_{,3} + \mathrm{dip}(\mathbf{y})$$

we obtain dipole axial part of the shift $\mathrm{dip}(h_{0A||B}\varepsilon^{AB}) = \dfrac{4\mathbf{s}}{r^3}$, and in particular for $\mathbf{s} = s\cos\theta$ we have $h_{0\phi} = -\dfrac{2s\sin^2\theta}{r}$ which has been proposed by Vishveshwara (eq. 5.2 in [3]).

B Reduction of the symplectic form

Let (p^{kl}, q_{kl}) denotes the Cauchy data on a hypersurface Σ. The (2+1)-splitting of the tensor q_{kl} gives the following components on a sphere: $\overset{2}{q} := \eta^{AB}q_{AB}$, q_{33} – scalars on S^2, q_{3A} – vector and $\overset{\circ}{q}_{AB} := q_{AB} - \frac{1}{2}\eta_{AB}\overset{2}{q}$ – symmetric traceless tensor on S^2. Similarly, we can decompose the tensor density p^{kl}. On each sphere $S(r)$ we can manipulate as follows

$$\int_V p^{kl}q_{kl} = \int_V p^{33}q_{33} + 2p^{3A}q_{3A} + \frac{1}{2}\overset{2}{p}\overset{2}{q} + \overset{\circ}{p}{}^{AB}\,\overset{\circ}{q}_{AB} =$$

$$= \int_V p^{33}q_{33} - 2(rp^{3A}{}_{||A})\,\overset{\circ}{\Delta}{}^{-1}(rq_{3A}{}^{||A}) - 2(rp^{3A||B}\varepsilon_{AB})\,\overset{\circ}{\Delta}{}^{-1}(rq_{3A||B}\varepsilon^{AB}) + \frac{1}{2}\overset{2}{p}\overset{2}{q} +$$

$$+2\int_V (r^2\varepsilon^{AC}\,\overset{\circ}{p}{}_A{}^B{}_{||BC})\,\overset{\circ}{\Delta}{}^{-1}(\overset{\circ}{\Delta} + 2)^{-1}(r^2\varepsilon^{AC}\,\overset{\circ}{q}{}_A{}^B{}_{||BC}) +$$

$$+2\int_V (r^2\,\overset{\circ}{p}{}^{AB}{}_{||AB})\,\overset{\circ}{\Delta}{}^{-1}(\overset{\circ}{\Delta} + 2)^{-1}(r^2\,\overset{\circ}{q}{}^{AB}{}_{||AB})$$

where we have used the following identities on a sphere

$$-\int_{S(r)} \pi^A v_A = (r\pi^A{}_{||A})\,\overset{\circ}{\Delta}{}^{-1}(rv^A{}_{||A}) + (r\pi^{A||B}\varepsilon_{AB})\,\overset{\circ}{\Delta}{}^{-1}(rv_{A||B}\varepsilon^{AB}) \tag{B.1}$$

and similarly for the traceless tensors we have

$$\int_{S(r)} \overset{\circ}{\pi}{}^{AB}\,\overset{\circ}{v}_{AB} = 2\int_{S(r)} (r^2\varepsilon^{AC}\,\overset{\circ}{\pi}{}_A{}^B{}_{||BC})\,\overset{\circ}{\Delta}{}^{-1}(\overset{\circ}{\Delta} + 2)^{-1}(r^2\varepsilon^{AC}\,\overset{\circ}{v}{}_A{}^B{}_{||BC}) +$$

$$+ 2\int_{S(r)}(r^2 \overset{\circ}{\pi}{}^{AB}{}_{||AB}) \overset{\circ}{\Delta}{}^{-1} (\overset{\circ}{\Delta} +2)^{-1}(r^2 \overset{\circ}{v}{}^{AB}{}_{||AB}) \tag{B.2}$$

The axial part of the quadratic form $\int_V p^{kl}q_{kl}$ we define as follows:

$$\text{axial part} = -2\int_V (rp^{3A||B}\varepsilon_{AB}) \overset{\circ}{\Delta}{}^{-1} (rq_{3A||B}\varepsilon^{AB})+$$

$$+2\int_V (r^2\varepsilon^{AC} \overset{\circ}{p}{}_A{}^B{}_{||BC}) \overset{\circ}{\Delta}{}^{-1} (\overset{\circ}{\Delta} +2)^{-1}(r^2\varepsilon^{AC} \overset{\circ}{q}{}_A{}^B{}_{||BC})$$

The remainder we define as a polar part. Moreover, from the axial part of the vector constraint (4.18):

$$(r^2p^{3A||B}\varepsilon_{AB}),_3 + r^2\varepsilon^{AC} \overset{\circ}{p}{}_A{}^B{}_{||BC} = 0 \tag{B.3}$$

we obtain

$$\text{axial part} = -2\int_V (rp^{3A||B}\varepsilon_{AB}) \overset{\circ}{\Delta}{}^{-1} (rq_{3A||B}\varepsilon^{AB})+$$

$$-2\int_V (r^2p^{3A||B}\varepsilon_{AB}),_3 \overset{\circ}{\Delta}{}^{-1} (\overset{\circ}{\Delta} +2)^{-1}(r^2\varepsilon^{AC} \overset{\circ}{q}{}_A{}^B{}_{||BC}) =$$

$$= -2\int_{\partial V} (r^2p^{3A||B}\varepsilon_{AB}) \overset{\circ}{\Delta}{}^{-1} (\overset{\circ}{\Delta} +2)^{-1}(r^2\varepsilon^{AC} \overset{\circ}{q}{}_A{}^B{}_{||BC})+$$

$$-2\int_V (r^2p^{3A||B}\varepsilon_{AB}) \overset{\circ}{\Delta}{}^{-1} [q_{3A||B}\varepsilon^{AB} - (\overset{\circ}{\Delta} +2)^{-1}(r^2\varepsilon^{AC} \overset{\circ}{q}{}_A{}^B{}_{||BC}),_3]$$

We can see the invariants in the volume term if we write dipole part separately

$$\text{dipole axial part} = -2\int_V \text{dip}(r^2p^{3A||B}\varepsilon_{AB}) \overset{\circ}{\Delta}{}^{-1}(q_{3A||B}\varepsilon^{AB}) \tag{B.4}$$

and finally the radiation axial part contains gauge-invariants in the volume term

$$\text{radiation axial part} = -2\int_{\partial V} (r^2p^{3A||B}\varepsilon_{AB}) \overset{\circ}{\Delta}{}^{-1} (\overset{\circ}{\Delta} +2)^{-1}(r^2\varepsilon^{AC} \overset{\circ}{q}{}_A{}^B{}_{||BC})+$$

$$- 2\int_V (r^2p^{3A||B}\varepsilon_{AB}) \overset{\circ}{\Delta}{}^{-1} (\overset{\circ}{\Delta} +2)^{-1}[(\overset{\circ}{\Delta} +2)q_{3A||B}\varepsilon^{AB} - (r^2\varepsilon^{AC} \overset{\circ}{q}{}_A{}^B{}_{||BC}),_3] \tag{B.5}$$

For the polar part we can use the rest of the vector constraints

$$\frac{r}{\sqrt{v}}(\sqrt{v}p^3{}_3)_{,3} + rp_{3A}{}^{||A} - \overset{2}{p} = 0 \tag{B.6}$$

$$(r^2p^{3A}{}_{||A}),_3 + (r^2 \overset{\circ}{p}{}^{AB}{}_{||AB}) + \frac{1}{2} \overset{\circ}{\Delta} \overset{2}{p} = 0 \tag{B.7}$$

and we can reduce partially polar part as follows

$$\text{polar part} = \int_V p^3{}_3q_3{}^3 - 2(rp^{3A}{}_{||A}) \overset{\circ}{\Delta}{}^{-1} (rq_{3A}{}^{||A}) + \frac{1}{2}\left(\frac{r}{\sqrt{v}}(\sqrt{v}p^3{}_3)_{,3} + rp^{3A}{}_{||A}\right) \overset{2}{q} +$$

$$-2\int_V \left[(r^2p^{3A}{}_{||A}),_3 + \frac{1}{2} \overset{\circ}{\Delta} \left(\frac{r}{\sqrt{v}}(\sqrt{v}p^3{}_3)_{,3} + rp_{3A}{}^{||A}\right)\right] \overset{\circ}{\Delta}{}^{-1} (\overset{\circ}{\Delta} +2)^{-1}(r^2 \overset{\circ}{q}{}^{AB}{}_{||AB}) =$$

$$= \int_{\partial V} rp^3{}_3 \left[\frac{1}{2} \overset{2}{q} -(\overset{\circ}{\Delta} +2)^{-1}(r^2 \overset{\circ}{q}{}^{AB}{}_{||AB})\right] - 2\int_{\partial V} (r^2p^{3A}{}_{||A}) \overset{\circ}{\Delta}{}^{-1} (\overset{\circ}{\Delta} +2)^{-1}(r^2 \overset{\circ}{q}{}^{AB}{}_{||AB})+$$

$$+\int_V rp_{3A}{}^{||A}\left[\frac{1}{2}\overset{2}{q}+2rv\overset{\circ}{\Delta}{}^{-1}(\overset{\circ}{\Delta}+2)^{-1}(r^2\overset{\circ}{q}{}^{AB}{}_{||AB}),_3-(\overset{\circ}{\Delta}+2)^{-1}(r^2\overset{\circ}{q}{}^{AB}{}_{||AB})-2\overset{\circ}{\Delta}{}^{-1}(rq^{3A}{}_{||A})\right]+$$

$$+\int_V p^3{}_3\left[q_3{}^3+\sqrt{v}(\overset{\circ}{\Delta}+2)^{-1}\left(\frac{r^3}{\sqrt{v}}\overset{\circ}{q}{}^{AB}{}_{||AB}\right),_3-\frac{\sqrt{v}}{2}\left(\frac{r}{\sqrt{v}}\overset{2}{q}\right),_3\right] \tag{B.8}$$

The above calculation shows that we should consider mono-dipole part separately. The monopole part is very simple

$$\mathrm{mon}(\int_V p^{kl}q_{kl})=\int_V\frac{1}{2}p_{33}\mathcal{B}^{-1}\mathrm{mon}(\zeta)+\frac{1}{2}\int_{\partial V}rp_3{}^3\mathrm{mon}(\overset{2}{q}) \tag{B.9}$$

where invariant ζ is defined as follows

$$\zeta:=\mathcal{B}\left[2q^{33}+2rq^{3A}{}_{||A}-rv\,\overset{2}{q},_3\right]+r^2\overset{\circ}{q}{}^{AB}{}_{||AB}-\frac{1}{2}(\overset{\circ}{\Delta}+2)\overset{2}{q}$$

Using (A.6), (A.7), (A.5) and integrating by parts we can also rewrite (from the beginning) dipole part in the following way

$$\text{dipole polar part}=\int_V p^{33}q_{33}-2(rp^{3A}{}_{||A})\overset{\circ}{\Delta}{}^{-1}(rq^{3A}{}_{||A})+\frac{1}{2}\overset{2}{p}\overset{2}{q}=$$

$$=\int_V\left[\frac{r^2\sqrt{v}}{6m}(\sqrt{v}\Pi)_{,3}-\frac{1}{2}\Pi\right]q_{33}-2\frac{r^2\sqrt{v}}{6m}(\sqrt{v}\Pi)_{,3}\overset{\circ}{\Delta}{}^{-1}(rq^{3A}{}_{||A})+\frac{1}{2}\left[\frac{r^3\sqrt{v}}{6m}(\sqrt{v}\Pi)_{,3}\right]_{,3}\overset{2}{q}=$$

$$=\int_{\partial V}\frac{r^3\sqrt{v}}{12m}(\sqrt{v}\Pi)_{,3}\overset{2}{q}+\frac{r^2}{6m}\Pi\left(rq^{3A}{}_{||A}+q^{33}-\frac{1}{2}vr\,\overset{2}{q}_{,3}\right)+$$

$$-\int_V\Pi\left\{\frac{1}{2}q_3^3+\frac{\sqrt{v}}{6m}\left[r^2\sqrt{v}\left(q_3^3-\frac{1}{2}r\,\overset{2}{q}_{,3}+rq_3{}^A{}_{||A}\right)\right]_{,3}\right\}=\int_{\partial V}\frac{r^3\sqrt{v}}{12m}(\sqrt{v}\Pi)_{,3}\overset{2}{q}+\frac{r^2v}{12m}\Pi Q=$$

$$=\int_{\partial V}rv\Pi\,\overset{\circ}{\Delta}{}^{-1}\left(\overset{\circ}{\Delta}+2-\frac{6m}{r}\right)^{-1}Q-r^2p^{3A}{}_{||A}\overset{\circ}{\Delta}{}^{-1}\overset{2}{q} \tag{B.10}$$

where here Π denotes only its dipole part and Π itself is defined by (A.1). We have also used scalar constraint (A.3), and finally the dipole polar part takes its boundary form (B.10).

The radiation polar part can be also reduced if we use scalar constraint (4.17) in two equivalent forms (motivated by (B.8))

$$\sqrt{v}\left(\frac{r^3}{\sqrt{v}}\overset{\circ}{q}{}^{AB}{}_{||AB}\right),_3+(\overset{\circ}{\Delta}+2)\left[q_3{}^3-\frac{\sqrt{v}}{2}\left(\frac{r}{\sqrt{v}}\overset{2}{q}\right),_3\right]=\sqrt{v}\left(\frac{r}{\sqrt{v}}\zeta\right),_3+\mathcal{B}\zeta \tag{B.11}$$

$$\frac{1}{2}\overset{\circ}{\Delta}(\overset{\circ}{\Delta}+2)\overset{2}{q}+2rv\left(r^2\overset{\circ}{q}{}^{AB}{}_{||AB}\right),_3-\overset{\circ}{\Delta}r^2\overset{\circ}{q}{}^{AB}{}_{||AB}-2(\overset{\circ}{\Delta}+2)\left(rq^{3A}{}_{||A}\right)=$$

$$=2v\sqrt{v}\left(\frac{r}{\sqrt{v}}\zeta\right),_3+2v\mathcal{B}\zeta-(\overset{\circ}{\Delta}+2)\mathcal{B}^{-1}\zeta \tag{B.12}$$

Inserting (B.11), (B.12) into (B.8) and integrating by parts we obtain

$$\text{radiation polar part}=\int_V p^3{}_3\left[q_3{}^3+\sqrt{v}(\overset{\circ}{\Delta}+2)^{-1}\left(\frac{r^3}{\sqrt{v}}\overset{\circ}{q}{}^{AB}{}_{||AB}\right),_3-\frac{\sqrt{v}}{2}\left(\frac{r}{\sqrt{v}}\overset{2}{q}\right),_3\right]+$$

$$+\int_V rp_{3A}{}^{||A}\left[\frac{1}{2}\overset{2}{q}+2rv\overset{\circ}{\Delta}{}^{-1}(\overset{\circ}{\Delta}+2)^{-1}\left(r^2\overset{\circ}{q}{}^{AB}{}_{||AB}\right),_3-(\overset{\circ}{\Delta}+2)^{-1}\left(r^2\overset{\circ}{q}{}^{AB}{}_{||AB}\right)-2\overset{\circ}{\Delta}{}^{-1}\left(rq^{3A}{}_{||A}\right)\right]+$$

$$+\int_{\partial V} rp^3{}_3 \left[\frac{1}{2}\overset{2}{q} - (\overset{\circ}{\Delta}+2)^{-1}\left(r^2\,\overset{\circ}{q}{}^{AB}{}_{||AB}\right)\right] - 2\int_{\partial V}\left(r^2p^{3A}{}_{||A}\right)\overset{\circ}{\Delta}{}^{-1}(\overset{\circ}{\Delta}+2)^{-1}\left(r^2\,\overset{\circ}{q}{}^{AB}{}_{||AB}\right) =$$

$$= \int_{\partial V} rp^3{}_3\frac{1}{2}\overset{2}{q} - r\Pi\,\overset{\circ}{\Delta}{}^{-1}(\overset{\circ}{\Delta}+2)^{-1}\left(r^2\,\overset{\circ}{q}{}^{AB}{}_{||AB}\right) + \int_V p^3{}_3(\overset{\circ}{\Delta}+2)^{-1}\left[\sqrt{v}\left(\frac{r}{\sqrt{v}}\zeta\right)_{,3} + \mathcal{B}\zeta\right] +$$

$$+\int_V rp^{3A}{}_{||A}\,\overset{\circ}{\Delta}{}^{-1}(\overset{\circ}{\Delta}+2)^{-1}\left[2\sqrt{v}\left(\frac{r}{\sqrt{v}}\zeta\right)_{,3} + 2\mathcal{B}\zeta - (\overset{\circ}{\Delta}+2)v^{-1}\mathcal{B}^{-1}\zeta\right] =$$

$$= \int_{\partial V} r\left[p^3{}_3 + 2\,\overset{\circ}{\Delta}{}^{-1}\left(rp^{3A}{}_{||A}\right)\right](\overset{\circ}{\Delta}+2)^{-1}\zeta + rp^3{}_3\frac{1}{2}\overset{2}{q} - r\Pi\,\overset{\circ}{\Delta}{}^{-1}(\overset{\circ}{\Delta}+2)^{-1}\left(r^2\,\overset{\circ}{q}{}^{AB}{}_{||AB}\right) +$$

$$+\int_V\left[-\frac{r}{\sqrt{v}}\left(\sqrt{v}\Pi\right)_{,3} + \mathcal{B}\Pi - (\overset{\circ}{\Delta}+2)\mathcal{B}^{-1}v^{-1}rp^{3A}{}_{||A}\right]\overset{\circ}{\Delta}{}^{-1}(\overset{\circ}{\Delta}+2)^{-1}\zeta =$$

$$= \int_{\partial V} rp^3{}_3\frac{1}{2}\overset{2}{q} + r\Pi\,\overset{\circ}{\Delta}{}^{-1}(\overset{\circ}{\Delta}+2)^{-1}\left(\zeta - r^2\,\overset{\circ}{q}{}^{AB}{}_{||AB}\right) + \int_V\left[2r^2\,\overset{\circ}{p}{}^{AB}{}_{||AB} + \mathcal{B}\Pi\right]\overset{\circ}{\Delta}{}^{-1}(\overset{\circ}{\Delta}+2)^{-1}\zeta \tag{B.13}$$

where we used equality

$$2r^2\,\overset{\circ}{p}{}^{AB}{}_{||AB} + \frac{r}{\sqrt{v}}\left(\sqrt{v}\Pi\right)_{,3} + (\overset{\circ}{\Delta}+2)\mathcal{B}^{-1}v^{-1}rp^{3A}{}_{||A} = 0$$

which is a simple consequence of the vector constraint (A.4), (B.6) and (B.7). It is easy to check that the volume terms in the final form of (B.5) and (B.13) contain demanded gauge-invariant result.

Let us summarize the result we have proved

$$\int_V p^{kl}q_{kl} = \text{monopole part} + \text{dipole part} + \text{radiation part}$$

monopole part is given by (B.9), the dipole part is a sum of (B.4) and (B.10):

$$\text{dipole part} = \text{dipole axial part} + \text{dipole polar part}$$

and radiation part is a sum of (B.5) and (B.13).

$$\text{radiation part in } V = \int_V\left[2r^2\,\overset{\circ}{p}{}^{AB}{}_{||AB} + \mathcal{B}\Pi\right]\overset{\circ}{\Delta}{}^{-1}(\overset{\circ}{\Delta}+2)^{-1}\zeta +$$

$$-2\int_V (r^2p^{3A||B}\varepsilon_{AB})\,\overset{\circ}{\Delta}{}^{-1}\left[q_{3A||B}\varepsilon^{AB} - (\overset{\circ}{\Delta}+2)^{-1}(r^2\varepsilon^{AC}\,\overset{\circ}{q}{}_A{}^B{}_{||BC})_{,3}\right]$$

$$\text{boundary terms} = \int_{\partial V} rv\Pi\,\overset{\circ}{\Delta}{}^{-1}\left(\overset{\circ}{\Delta}+2-\frac{6m}{r}\right)^{-1}Q - r^2p^{3A}{}_{||A}\,\overset{\circ}{\Delta}{}^{-1}\,\overset{2}{q} +$$

$$-2\int_{\partial V}(r^2p^{3A||B}\varepsilon_{AB})\,\overset{\circ}{\Delta}{}^{-1}(\overset{\circ}{\Delta}+2)^{-1}(r^2\varepsilon^{AC}\,\overset{\circ}{q}{}_A{}^B{}_{||BC})$$

C (2+1)-decomposition of the gauge for lapse and shift

The gauge transformation (2.18) splits according to the (2+1)-decomposition and we obtain the following gauge transformation law which acts on lapse and shift:

$$h_{00} \rightarrow h_{00} + 2\dot{\xi}_0 - \frac{2m}{r^2}\xi_3 \tag{C.1}$$

$$h_{0A} \rightarrow h_{0A} + \xi_{0,A} + \xi_{A,0} \tag{C.2}$$

$$h_{03} \rightarrow h_{03} + \dot{\xi}_3 - v\xi^0_{,3} \tag{C.3}$$

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